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An exact analytical method for inferring the law of gravity from the macroscopic dynamics: Spherical mass distribution with exponential density

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Abstract. We consider the gravitational potential and the gravitational rotation field generated by an spherical mass distribution with exponential density, when the force between any two mass elements is not the usual Newtonian one, but some general central force. We invert the usual integral relations and obtain the elemental interaction (between two point-like masses) as a function of the macroscopic gravitational field (the one generated by the distribution). Thus, we have a direct way for testing the possibility of finding a correction to the Newtonian law of gravity that can explain the observed dynamics at large scales without the need of dark matter. We show that this method can be used even in the case of spiral galaxies with a good level of confidence.

Key words: Gravitation – Galaxies: kinematics and dynamics – Methods: analytical – *(Cosmology:)* dark matter

1. Introduction

The dynamical analysis of rotation curves of galaxies, binary galaxies, clusters of galaxies and the structures known at large scales show large discrepancies between the observed behaviour and the one expected from the application of General Relativity and its Newtonian limit to the visible mass. This disagreement has led many astrophysicists to believe in the existence of a large amount of non-visible matter and is, thus, commonly known as the *Dark matter problem*.

In spite of this, there is no direct evidence for the validity of either Einstein's General Relativity or Newtonian gravity at scales much larger than those of the Solar system (See, e.g., Will 1993). There is therefore no experimental or observational reason to ascertain that unmodified General Relativity holds at larger distances. This leads us to think that we should be open to the possibility that it had to be revised (perhaps in the same spirit as Newton's law had to be modified for strong fields and large velocities).

In this paper we consider the possibility that Newton's law of gravity is just a good approximation at short distances of a more general expression for the force. It is interesting to identify which, if any, extensions of the usual inverse square law are compatible with the dynamics observed at large scales.

Work along these lines has already been done (Tolhine 1983, Kuhn & Kruglyak 1987, Mannheim & Kazanas 1989) assuming a specific functional form for the force, and then evaluating the field generated by a mass distribution (for instance, a galaxy) by performing the corresponding three-dimensional integrals. We present and work out a method that allows us to follow the inverse methodology, that is, to infer, directly from observations, the phenomenological law of gravity that is able to generate a given macroscopic gravitational field. We do it for the case of a spherical mass distribution with exponential density.

In Section 2 we give the general definitions that will be used later in Section 3 for the case of spherical symmetry with exponential density. In Section 4 we show the mathematical basis underlying the results presented in Section 3. Finally we offer some conclusions. In Appendix A we consider the possibility of applying these results as an approximation

to the study of spiral galaxies overcoming the fact that they do not show spherical symmetry. In Appendix B we list some of the mathematical identities used in Section 4.

2. General definitions

Let us assume that the gravitational potential generated by a point-like mass does not correspond to the usual Newtonian form but can be written in terms of a function $g(r)$ that describes the deviation from the Newtonian law, that is,

$$\phi(r) \equiv -\frac{G_0 m_1 m_2}{r} g(r). \tag{1}$$

where $\phi(r)$ is the gravitational potential experienced by two point-like particles of masses m_1 and m_2 separated by a distance r and G_0 is the Newton's constant.

This modification could, for example, be due to the many body nature of the mass distribution making up the galaxy, quantum corrections, a relativistic theory different from General Relativity... This is irrelevant in what follows.

The force per unit mass is, by definition, the gradient of the potential,

$$\boldsymbol{F}(r) \equiv -\frac{G_0 m_1 m_2}{r^2} g_{\rm eff}(r) \frac{\boldsymbol{r}}{r}, \tag{2}$$

where we have introduced

$$g_{\rm eff}(r) \equiv g(r) - r g'(r). \tag{3}$$

In this way, to find the total potential or the total force generated by a mass distribution Ω with density $\rho(\boldsymbol{r})$, one must integrate over the volume spanned by Ω to get:

$$\Phi(\boldsymbol{R}) = -G_0 \int\int\int_\Omega d^3\boldsymbol{r} \;\; \frac{g(|\boldsymbol{R}-\boldsymbol{r}|)}{|\boldsymbol{R}-\boldsymbol{r}|} \;\; \rho(\boldsymbol{r}) \tag{4}$$

for the potential experienced by a point mass at a distance R from the centre of Ω, and

$$\boldsymbol{F}(\boldsymbol{R}) = -G_0 \int\int\int_\Omega d^3\boldsymbol{r} \;\; \frac{g_{\rm eff}(|\boldsymbol{R}-\boldsymbol{r}|)}{|\boldsymbol{R}-\boldsymbol{r}|^2} \;\; \frac{\boldsymbol{R}-\boldsymbol{r}}{|\boldsymbol{R}-\boldsymbol{r}|} \;\; \rho(\boldsymbol{r}) \tag{5}$$

for the force.

In the case that the gravitational potential is only a function of the distance to the centre of the distribution, it is convenient to introduce two new functions $\Psi(R)$ and $\Psi_{\rm eff}(R)$ such that:

$$\Phi(R) \;\equiv\; -\;\frac{G_0 M_{\rm tot}}{R}\;\Psi(R), \tag{6}$$

$$\boldsymbol{F}(R) \;\equiv\; -\frac{G_0 M_{\rm tot}}{R^2}\;\Psi_{\rm eff}(R)\;\frac{\boldsymbol{R}}{R}, \tag{7}$$

and the rotation velocity of a test particle in a circular orbit bound to the distribution will be:

$$V^2_{\rm rot}(R) \;\equiv\; \frac{G_0 M_{\rm tot}}{R}\;\Psi_{\rm eff}(R) \tag{8}$$

where the auxiliary functions $\Psi_{\rm eff}(R)$ and $\Psi(R)$ satisfy the following functional relationship:

$$\Psi_{\rm eff}(R) = \Psi(R) - R\Psi'(R), \tag{9}$$

Our goal is to design a procedure where, assuming that $\boldsymbol{F}(R)$ is known (say from observation of the rotation velocity) for all values of R, we obtain a $g_{\rm eff}(r)$ that generates the given rotation velocity. Or, in other words, given the potential as inferred from observations we want to find which $g(r)$ could have generated it. Actually, what we will find are $g(r)$ and $g_{\rm eff}(r)$ as functions of $\Psi_{\rm eff}(R)$ and $\Psi(R)$ respectively.

3. Spherical mass distribution with exponential density

In view of the applications that we have in mind, we study a spherically symmetric distribution with an exponentially decaying density:

$$\rho(r) \equiv \rho_0 \; e^{-\alpha r}. \tag{10}$$

Our ultimate goal is to find a method to study the discrepancies between the observed rotation curves of spiral galaxies and the curves predicted by using Newton's law of gravity. The luminosity profile of many spiral galaxies can be well fitted assuming that the density of luminous matter decreases exponentially with distance from the centre of the galaxy (Kent 1987). This is the reason why we are interested in studing such a density function, even though spiral galaxies are not spherical.

Using Eqs. (6), (7) and (10), in Eqs. (4) and (5), and considering spherical symmetry, the two problems sketched in Section 2 can be conveniently recast in the form of two integral equations:

(i) Given $\Psi(R)$, find a function $g(r)$ that satisfies the equation:

$$\int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{g(\sqrt{R^2+r^2-2Rrcos\theta})}{\sqrt{R^2+r^2-2Rrcos\theta}} r^2 sin\theta e^{-\alpha r} = \frac{8\pi}{\alpha^3}\frac{\Psi(R)}{R} \tag{11}$$

and

(ii) Given $\Psi_{\rm eff}(R)$, find a function $g_{\rm eff}(r)$ such that:

$$\int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{g_{\rm eff}(\sqrt{R^2+r^2-2Rrcos\theta})}{(R^2+r^2-2Rrcos\theta)^{\frac{3}{2}}}(R-rcos\theta) r^2 sin\theta e^{-\alpha r} = \frac{8\pi}{\alpha^3}\frac{\Psi_{\rm eff}(R)}{R^2}. \tag{12}$$

In the next section the solution to these integral equations will be described in detail. The results can be summarized as:

(i) Potential problem (*viz.* Eqs. (6) and (11))

In this case, the exact solution to the problem is

$$g(x) = \Psi(x) - \frac{2}{\alpha^2}\Psi''(x) + \frac{1}{\alpha^4}\Psi^{(iv)}(x) \tag{13}$$

where the function Ψ has the following behaviour at the origin:

$$\Psi(0) \;=\; \Psi''(0) \;=\; 0. \tag{14}$$

(ii) Force and velocity problem (*viz.* Eqs. (7), (8) and (12)).

Here, the exact solution is given by the following expression:

$$g_{\rm eff}(x) = \Psi_{\rm eff}(x) - \frac{2}{\alpha^2}\Psi''_{\rm eff}(x) + \frac{1}{\alpha^4}\Psi^{(iv)}_{\rm eff}(x) + \frac{4}{\alpha^2 x}\Psi'_{\rm eff}(x) - \frac{4}{\alpha^4 x^4}[2x\Psi'_{\rm eff}(x) - 2x^2\Psi''_{\rm eff}(x) + x^3\Psi'''_{\rm eff}(x)] \tag{15}$$

The behaviour of Ψ at the origin is as follows:

$$\Psi_{\rm eff}(0) \;=\; \Psi'_{\rm eff}(0) \;=\; \Psi''_{\rm eff}(0) \;=\; 0. \tag{16}$$

The behaviours at the origin just tell us that $\Psi_{\rm eff}(R) \propto R^3$ for $R \sim 0$, and thus, $V_{\rm rot}(R) \propto R$ for $R \sim 0$, which is in fact in good agreement with the observations (as the observed rotation curves are usually well fitted in the inner regions by a straight line) and, for a non-Newtonian gravity point of view, it is also in agreement with the fundamental experimental constrain that for short distances the gravitational interaction must be well described by a Newtonian limit.

4. Mathematical development

First, we will show the solution to what we call *the potential problem*, that is, how to go from Eq. (11) to Eqs. (13) and (14). Later we will use these results to solve *the force problem*, i.e, how to go from Eq. (12) to Eqs. (15) and (16).

4.1. The potential problem.

In order to go from equation Eq. (11) to Eqs. (13) and (14), we will need to use several mathematical identities. For convenience, these are listed in Appendix B.

Inserting Eqs. (B1), (B3), (B5), (B6) and (B7) into Eq. (11) we can write $\Psi(R)$ as:

$$\Psi(R) = -\frac{\alpha^3}{\pi} \int_0^\infty dp \frac{\hat{g}_s(p)}{p} \int_0^\infty dr e^{-\alpha r} r \sin pr \sin pR = -\frac{\alpha^3}{\pi} \frac{d}{d\alpha} \int_0^\infty dp \frac{\hat{g}_s(p)}{p^2+\alpha^2} \sin pR \tag{17}$$

Then, we can apply Eq. (B2) and invert the Fourier transform to obtain, after some straightforward calculations, the following more useful form:

$$\begin{aligned}\Psi(R) = -\tfrac{\alpha}{4}\ \Big\{ &(1-\alpha R)e^{\alpha R}\left[\int_0^R dx\ g(x)\ e^{-\alpha x}\ -\ \int_0^\infty dx\ g(x)\ e^{-\alpha x}\right] - \\ &-(1+\alpha R)e^{-\alpha R}\left[\int_0^R dx\ g(x)\ e^{\alpha x}\ -\ \int_0^\infty dx\ g(x)\ e^{-\alpha x}\right] + \\ &+\alpha e^{\alpha R}\left[\int_0^R dx\ g(x)\ x\ e^{-\alpha x}\ -\ \int_0^\infty dx\ g(x)\ x\ e^{-\alpha x}\right]\ + \\ &+\ \alpha e^{-\alpha R}\left[\int_0^R dx\ g(x)\ x\ e^{\alpha x}\ +\ \int_0^\infty dx\ g(x)\ x\ e^{-\alpha x}\right]\quad \Big\}\end{aligned} \tag{18}$$

In order to simplify these expressions, we introduce an auxiliary function $\psi(x)$ that makes the integrals exact:

$$g(x)\ \equiv\ \psi(x) - \frac{2}{\alpha^2}\ \psi''(x) + \frac{1}{\alpha^4}\ \psi^{iv}(x) \tag{19}$$

We can insert Eq (19) into Eq. (18) and, upon integration by parts, we get:

$$\Psi(R) = \psi(R) - (1+\frac{\alpha R}{2})e^{-\alpha R}\psi(0) + \frac{R}{2\alpha^2}e^{-\alpha R}\psi''(0) \tag{20}$$

where ψ is a solution to the ordinary differential Eq. (19) that satisfies the conditions of being an analytic function at $x=0$, and

$$\lim_{x\to\infty}\ \psi^{(k)}(x)\ x\ \exp(-\alpha x) = 0\ ;\ k = 0,1,2,3 \tag{21}$$

where $\psi^{(k)}(x)$ stands for $\psi(x)$ and its three first derivatives.

These conditions are easily fulfilled in all the cases of interest. Actually, the analiticity is satisfied in the Newtonian limit, that is the behaviour that we expect to recover at $R\sim 0$. Although it is possible to artificially build a *pathological* $\psi(x)$ such that it can represent a physical system without satisfying Eq. (21), it can be seen that almost every ψ function that does not satisfy it corresponds to a rotation velocity that grows almost exponentially with the distance, which clearly seems to contradict the observations.

Actually, it is straightforward to see that, provided $\psi(R)$ is a solution to Eq. (19), then $\Psi(R)$ is also a solution to the same equation. Moreover, the terms proportional to $\psi(0)$ and $\psi''(0)$ in Eq. (20) assure that Ψ and its second derivative are both zero at the origin. Taking all of this into consideration, we finally obtain that:

$$g(x) = \Psi(x) - \frac{2}{\alpha^2}\Psi''(x) + \frac{1}{\alpha^4}\Psi^{(iv)}(x), \tag{22}$$

$$\Psi(0)\ =\ \Psi''(0)\ = 0. \tag{23}$$

4.2. The force and the rotation velocity.

Once $\Psi(R)$ is known we can calculate $\Psi_{\rm eff}(R)$ using Eq. (9). Equivalently, once $g(r)$ is known, $g_{\rm eff}(r)$ can be obtained through Eq. (3). Using these two equations together with Eq. (22), and after some straightforward calculations, we can find a direct relation between $g_{\rm eff}$ and $\Psi_{\rm eff}$:

$$g_{\rm eff}(x) = \Psi_{\rm eff}(x) - \frac{2}{\alpha^2}\Psi''_{\rm eff}(x) + \frac{1}{\alpha^4}\Psi^{iv}_{\rm eff}(x) + \frac{4}{\alpha^2 x}\Psi'_{\rm eff}(x) - \frac{4}{\alpha^4 x^4}[2x\Psi'_{\rm eff}(x) - 2x^2\Psi''_{\rm eff}(x) + x^3\Psi'''_{\rm eff}(x)] \tag{24}$$

From the behaviour of Ψ at the origin (Eq. (23), and Eq. (9)), it is easy to see that, at the origin, $\Psi_{\rm eff}$ will satisfy:

$$\Psi_{\rm eff}(0)\ =\ \Psi'_{\rm eff}(0)\ =\ \Psi''_{\rm eff}(0)\ =\ 0. \tag{25}$$

5. Discusion and conclusions

We have found the *exact* solution to the problem of inverting the integral relation between the elemental law of gravity and the gravitational field generated by a spherical mass distribution. We have arrived at a direct and simple expression which can be useful in order to infer the law of gravity that could explain the large scale gravitational behaviour if we had good data on rotation curves of spherical galaxies with an exponential density profile. However, most of the observational data available on rotation curves are for spiral galaxies. Despite the fact, shown in Appendix A, that this formalism is a good approximation to study this class of galaxies if $g_{\rm eff}$ is a growing function of r, we believe that it is better to study the case of a thin disk distribution that is a much better approximation to the real morphology of spiral galaxies. In two forthcoming publications we will give a similar solution for a thin-disk distribution (Rodrigo-Blanco 1996) and we will apply it to real galaxies (Rodrigo-Blanco & Pérez-Mercader 1996).

A. Appendix: Gravitational difference between a disk and a sphere

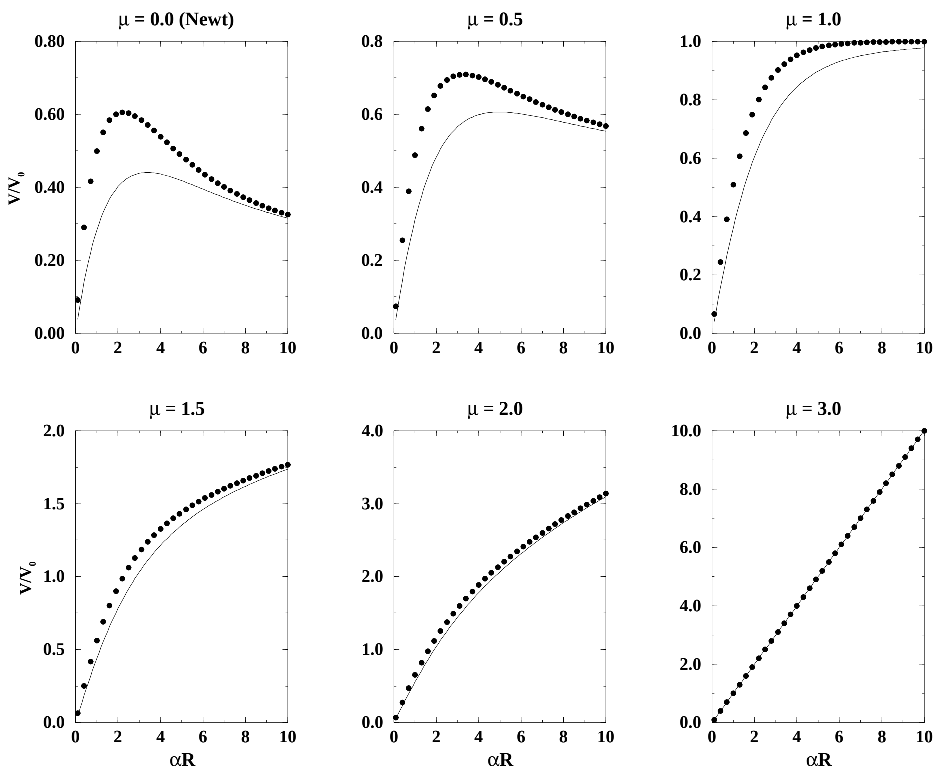

Fig. 1. Rotation velocities for a sphere (solid line) and for a disk (dots) when $g_{\rm eff}(r) \equiv \left(\frac{r}{a}\right)^{\mu}$ for some values of μ. In every case, for the sake of clarity, the velocities are normalized by dividing by the appropiate constant $\frac{G_0 M_{\rm tot}\alpha}{(\alpha a)^{\mu}}$.

At first sight, we would think that we cannot apply the results of the previous sections to spiral galaxies, because they have been obtained assuming spherical symmetry, and spiral galaxies are disk-shaped, not spherical. In spite of this, it seems logical that the difference between the gravitational field generated by a sphere and the one generated by a disk becomes negligible at very large distances (here "large distances" means large compared with some typical length scale for the distribution; it could be, for instance, α^{-1}, i.e, the exponential length scale of the mass distribution). Furthermore, when we consider the effect of an increasing $g_{\rm eff}(r)$, it is evident that the meaning of what is a "large distance" changes as we change the functional form of $g_{\rm eff}(r)$. That is, the faster $g_{\rm eff}(r)$ grows as a function of r, the smaller the distance at which a sphere is indistinguishable from a disk becomes, when considered from the gravitational point of view.

In order to obtain a more quantitative idea of the gravitational difference between a sphere and a disk, when $g_{\rm eff}$ is a growing function of its argument, we have chosen a parametric family of $g_{\rm eff}$'s given by:

$$g_{{\rm eff},\mu}(r) \equiv \left(\frac{r}{a}\right)^{\mu} \qquad \text{(A1)}$$

where μ parameterizes how fast $g_{\rm eff}$ grows.

Then we have calculated the rotation velocity performing the three-dimensional integral numerically for several values of μ. We have done it both for the case of spherical symmetry, and for the case of cylindrical symmetry. We have assumed for the disk a small thickness of $h = \alpha^{-1}/6$, and the rotation velocity is calculated in the plane of the disk.

In figure (1), we have plotted both velocities for a disk as well as for a sphere for some values of μ. In each case we have normalized the solutions dividing by a convenient constant V_0 defined as:

$$V_0 \equiv \frac{G_0 M_{\rm tot}\alpha}{(\alpha a)^{\mu}} \qquad \text{(A2)}$$

It can be seen that both curves tend to merge as μ increases.

However, we would like to have a more quantitative way of describing the difference between both curves as a function of μ. In order to do this, we define a quantity σ_D^2 (see below for its meaning) as follows:

$$\sigma_D^2(\mu) \equiv \frac{1}{N}\sum_{\rm i=1}^{N} \frac{(V_{{\rm D},\mu}(r_i) - V_{{\rm S},\mu}(r_i))^2}{V_{{\rm D},\mu}^2(r_i)} \qquad \text{(A3)}$$

Here the subscripts D and S stand for 'disk' and 'sphere', respectively. We sum over r_i, which are the points where the integrals are calculated. The total number of points for each value of μ is $N = 100$.

Because of the way it is defined, $\sigma_D^2(\mu)$ is a measure of the mean square error that we make in the rotation velocity if we consider a sphere instead of the real disk, for each value of μ. We thus aim for a value as small as posible for σ_D^2. In Figure (2), we plot the value of σ_D^2 versus μ. Once again, it can be seen that the larger the value of μ is, the smaller the difference.

Considering what these plots mean, we see that our mathematical results can be used in the case of spiral galaxies, with a good level of confidence, provided $g_{\rm eff}(r)$ grows fast enough with r. Actually, the most popular corrections to $g_{\rm eff} = 1$ that can be found in the literature are $g_{\rm eff} \propto r$, (see Tohline 1983 and Khun & Kruglyak 1987), and $g_{\rm eff} \propto r^2$, (see Mannheim & Kazanas 1989). It can be seen in the figures that the approximation is quite good in both cases. Although the results obtained in this way are not exact, they give a qualitatively correct picture. However, it will be better to use the solution to the problem in the case of disk symmetry itself, as we will do in a forthcoming publication (Rodrigo-Blanco 1996).

B. Appendix: Mathematical identities

1. Fourier sine transform

$$g(\sqrt{r^2+R^2-2rR\cos\theta}) \equiv \frac{2}{\pi}\int_0^{\infty} \hat{g}_s(p)\sin(p\sqrt{r^2+R^2-2rR\cos\theta})dp \qquad \text{(B1)}$$

$$\hat{g}_s(p) = \int_0^{\infty} g(t)\sin ptdt \qquad \text{(B2)}$$

2. Addition theorem for Bessel functions (see Gradshteyn 1980).

$$\frac{Z_\nu(m\omega)}{\omega^\nu} = \frac{2^\nu}{m^\nu}\Gamma(\nu)\sum_{\rm k=0}^{\infty}(\nu+k)\frac{J_{\nu+{\rm k}}(m\rho)}{\rho^\nu}\frac{J_{\nu+{\rm k}}(mr)}{r^\nu}C_k^\nu(\cos\theta) \qquad \text{(B3)}$$

where:

$$\begin{cases} \omega \equiv \sqrt{r^2+\rho^2-2r\rho\cos\theta} \\ \rho < r \\ C_k^\nu \equiv \text{Gegenbauer Polynomials.} \end{cases} \qquad \text{(B4)}$$

Using Eq. (B3), together with:

$$\sin(mz) = \sqrt{\frac{\pi m z}{2}}J_{1/2}(mz) \qquad \text{(B5)}$$

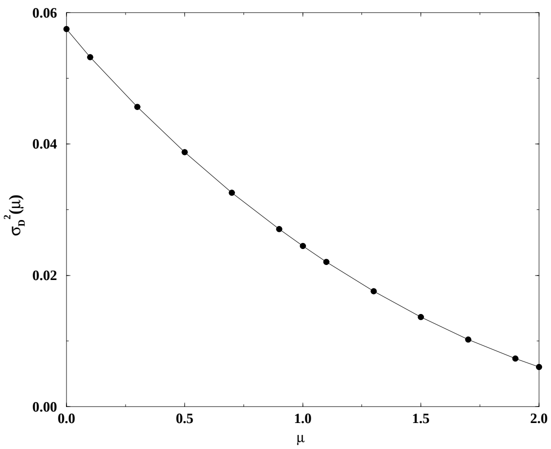

Fig. 2. Mean square error (σ_{D}^2) in the velocity as a function of μ when a sphere is considered instead of a disk, and using $g_{eff}(r) \equiv \left(\frac{r}{a}\right)^{\mu}$. The exact meaning of σ_{D}^2 is explained in the text.

we get:

$$\frac{\sin p\sqrt{r^2+R^2-2rR\cos\theta)}}{\sqrt{r^2+R^2-2rR\cos\theta}} = \frac{\pi}{2\sqrt{Rr}}\sum_{\mathrm{k}=0}^{\infty}(2k+1)J_{\mathrm{k}+\frac{1}{2}}(pr)J_{\mathrm{k}+\frac{1}{2}}(pR)P_k(\cos\theta) \tag{B6}$$

3. Orthogonality of Legendre Polynomials.

$$\int_0^{\pi} d\theta P_k(\cos\theta)\sin\theta = 2\delta_{\mathrm{k},0} \tag{B7}$$

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